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High temperature dimensional reduction in Snyder space

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A B S T R A C T

In this paper, we formulate the statistical mechanics in Snyder space that supports the existence of a minimal length scale. We obtain the corresponding invariant Liouville volume which properly determines the number of microstates in the semiclassical regime. The results show that the number of accessible microstates drastically reduces at the high energy regime such that there is only one degree of freedom for a particle. Using the Liouville volume, we obtain the deformed partition function and we then study the thermodynamical properties of the ideal gas in this setup. Invoking the equipartition theorem, we show that 2/3 of the degrees of freedom freeze at the high temperature regime when the thermal de Broglie wavelength becomes of the order of the Planck length. This reduction of the number of degrees of freedom suggests an effective dimensional reduction of the space from 3 to 1 at the Planck scale.

1. Introduction

While general relativity and quantum mechanics are successful in their applicability domains, it seems that there is a fundamental incompatibility between them in order to find the so-called quantum theory of gravity. Such a theory, not completely formulated so far, would reasonably describe the structure of spacetime at the Planck scale where both of the gravitational and quantum mechanical effects become important. Despite the fact that there is no unique approach to quantum gravity, existence of a universal minimum measurable length, preferably of the order of the Planck length $l_{\rm Pl} \sim 10^{-33}$ m, is a common feature of quantum gravity candidates such as string theory and loop quantum gravity [1,2]. It is then widely believed that a non-gravitational theory which includes a universal minimal length scale would appear at the flat limit of quantum gravity. Therefore, many attempts have been done in order to take into account a minimal length scale in the well-known non-gravitational theories such as quantum mechanics and special relativity. The generalized uncertainty principle is investigated in the context of the string theory that supports the existence of a minimal length as a nonzero uncertainty in position measurement [3]. Quantum field theories turn out to be naturally ultraviolet-regularized in this setup [4]. Inspired by the seminal work of Snyder in 1947 who was formulated a Lorentz-invariant noncommutative spacetime [5], a phase space with noncanonical symplectic structure is formulated in the non-relativistic limit [6]. At the quantum level, this deformed phase space leads to the modified uncertainty relation which is very similar to one arises from the string theory motivations [6]. Furthermore, recently, polymer quantum mechanics was suggested in the symmetric sector of loop quantum gravity which supports the existence of a minimal length scale known as the polymer length scale [7]. Also, the doubly special relativity theories are investigated in order to take into account a minimal observer-independent length scale in special relativity [8]. Appearance of curved energy–momentum space is the direct consequence of the doubly special relativity theories [9] and, interestingly, the Snyder noncommutative spacetime could be also realized in this setup by a relevant gauge fixing process [10].

Apart from the details of the above mentioned phenomenological models as a flat limit for quantum gravity, all of them suggest the deformation to the density of states at the high energy regime which in turn leads to the nonuniform measure over the set of microstates. Indeed, in these setups, the number of accessible microstates will be reduced at the high energy regime due to the existence of a minimal length as an ultraviolet cutoff for the system under consideration. Reduction of the number of degrees of freedom, however, immediately suggests an effective dimensional reduction of the space. This consequence seems to be a general feature of quantum gravity which may also open new window for the statistical origin of black holes thermodynamics [11]. Thermodynamics of black holes is widely studied in the frameworks of phenomenological quantum gravity models such as noncommutative space [12], generalized uncertainty principle [13] and polymer

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quantization scenario [14]. The reduction of the number of accessible microstates due to the universal quantum gravitational effects would also significantly change the thermodynamical properties of any physical system at the high temperature regime. Therefore, quantum gravity effects on the thermodynamics of various statistical systems are widely studied in different contexts [15]. For the special case of the ideal gas, it is natural to expect that the quantum gravity effects would become important at the high temperature regime, when the corresponding thermal de Broglie wavelength $\lambda = \sqrt{\frac{2\pi}{mT}}\,\hbar$ becomes of the order of the Planck length $l_{\rm Pl} = \sqrt{\hbar G}$, where m is the particles' mass and T denotes the temperature.[1] The associated thermodynamical properties then will be significantly modified in this regime. Thermodynamics of the ideal quantum gases in noncommutative space is studied in Refs. [16, 17] and for the case of the effects that arise from the generalized uncertainty principle see Ref. [18]. Thermodynamical properties of the ideal gas in polymerized phase space, as a classical limit of a polymer quantum mechanics, are also studied in Refs. [19–21]. For the case of the relativistic ideal gases in doubly special relativity framework see Refs. [22,23]. Motivated by the above stated issues, in this paper we study the thermodynamical properties of the ideal gas in Snyder space.

The structure of the paper is as follows: In Section 2, the statistical mechanics in the Snyder space is formulated and the corresponding partition function is found. Using the partition function, thermodynamics of the ideal gas is studied in Section 3. Section 4 is devoted to the summary and conclusions.

2. Statistical mechanics in Snyder space

The kinematics and dynamics of a classical system on the phase space provide a suitable framework for formulating the statistical mechanics in the semiclassical regime. The key quantity is the Liouville volume that determines the density of states from which all the thermodynamical properties of a system could be achieved. In this section, using the symplectic geometry, we formulate the statistical mechanics in Snyder space.

2.1. Kinematics and dynamics

Inspired by the seminal work of Snyder on noncommutative spacetime [5], the associated deformed phase space is formulated which also supports the existence of a minimal length [6]. A phase space naturally admits symplectic structure and therefore is a symplectic manifold. Suppose that (Γ, ω) to be a Snyder-deformed phase space with ω as the associated symplectic structure which is a closed nondegenerate 2-form on Γ. The local form of the symplectic structure in Snyder model is given by [24]

$$\omega = dq^i \wedge dp_i - \frac{1}{2}\, d(q^i p_i) \wedge d\ln\Big[1 + \beta^2 p^2\Big]\,, \tag{1}$$

where q^i and p_i are the position and momentum coordinates of a particle with $i, j = 1, \ldots, 3$ and $p^2 = \delta^{ij} p_i p_j$. β is the deformation parameter with dimension of length which is usually taken to be of the order of the Planck length as $\beta = \beta_0\, l_{\rm Pl}$, where $\beta_0 = \mathcal{O}(1)$ is the dimensionless numerical constant that should be fixed only with experiment [25]. Taking the low energy limit $\beta \to 0$ in the relation (1), the standard well-known canonical form of the symplectic structure could be recovered.

Since the symplectic structure is nondegenerate by definition, one can assign a unique vector field $\mathbf{x}_f$ to any function f on Γ as $\omega(\mathbf{x}_f) = df$. The Poisson bracket for two real-valued functions is defined as

$$\{f, g\} = \omega(\mathbf{x}_f, \mathbf{x}_g)\,. \tag{2}$$

From the above definition, it is straightforward to show that the symplectic structure (1) generates the following noncanonical Poisson algebra

$$\{q^i, q^j\} = \beta^2 J^{ij}, \qquad \{q^i, p_j\} = \delta^i_j + \beta^2 p^i p_j, \qquad \{p_i, p_j\} = 0, \tag{3}$$

where $J_{ij} = q_i p_j - q_j p_i$ is the generator of the rotation group in three dimensions with $q_i = \delta_{ij} q^j$. The symplectic structure (1) or equivalently the Poisson algebra (3) properly defines the kinematics of the phase space Γ in Snyder model.

The dynamics of the system will be determined by specifying a Hamiltonian function H as the generator of time evolution of the system. The Hamiltonian system on the phase space is then defined by the triplet (Γ, ω, H) and the dynamical evolution of the system is governed by the equation

$$\omega(\mathbf{x}_H) = dH\,, \tag{4}$$

where $\mathbf{x}_H$ is the Hamiltonian vector field and it's integral curves are nothing but the Hamilton's equations in this setup (see Ref. [24] for more details).

Furthermore, the natural volume on the phase space is the Liouville volume which for a $2n$-dimensional phase space is defined as

$$\omega^n = \frac{1}{n!}\, \omega \wedge \ldots \wedge \omega \qquad (n \text{ times})\,. \tag{5}$$

The Liouville volume for a particle in Snyder-deformed phase space then could be easily obtained by substituting the symplectic structure (1) into the definition (5) which gives

$$\omega^3 = dq^1 \wedge dq^2 \wedge dq^3 \wedge \frac{dp_1 \wedge dp_2 \wedge dp_3}{\big(1 + \beta^2 p^2\big)}\,. \tag{6}$$

The phase space (Liouville) volume determines the density of states and then the number of accessible microstates for a statistical system. It is important to check the verification of the Liouville theorem for the Snyder measure (6) in order to formulate the statistical mechanics in Snyder-deformed phase space. The Liouville theorem states that the Liouville volume is invariant under the time evolution of the system

$$\frac{d\omega^n}{dt} = \frac{\partial \omega^n}{\partial t} + \mathcal{L}_{\mathbf{x}_H} \omega^n = 0\,, \tag{7}$$

where $\mathcal{L}_{\mathbf{x}_H}$ denotes the Lie derivative with respect to $\mathbf{x}_H$. The relation (7) can be traced back to the facts that ω^n is not explicitly time-dependent, $\mathcal{L}_{\mathbf{x}_H}\omega^n = n(\mathcal{L}_{\mathbf{x}_H}\omega) \wedge \omega^{n-1}$ and $\mathcal{L}_{\mathbf{x}_H}\omega = d(\omega(\mathbf{x}_H)) + (d\omega)(\mathbf{x}_H) = 0$, where we have used the equation (4) as $d(\omega(\mathbf{x}_H)) = d^2 H = 0$ and the closure of the symplectic structure $d\omega = 0$. The result (7) for the Snyder measure (6) is essential for us to formulate the statistical mechanics in Snyder space.

2.2. Number of microstates

Before obtaining the partition function, from which all the thermodynamical properties of a system can be obtained, we first make a qualitative discussion about the number of microstates in Snyder-deformed phase space regardless of the ensemble density and the Hamiltonian function.

The number of microstates for a single-particle phase space is given by

[1] We work in units $k_B = 1 = c$, where k_B and c are the Boltzmann constant and speed of light in vacuum respectively.

$$N_m = \frac{1}{h^3} \int\limits_{p \le p_*} \omega^3 \,, \tag{8}$$

where we have considered a region of phase space in which the condition $p \le p_*$, with $p = \sqrt{\delta^{ij} p_i p_j}$ is satisfied in order to have a finite number of microstates.

The quantity (8) for a system with non-deformed phase space is $N_m = (4\pi V/h^3) \int_0^{p_*} p^2 dp = (4\pi V/3h^3)\, p_*^3$, where V is the spatial volume of the system under consideration. For the case of the Snyder-deformed phase space, substituting the Liouville volume (6) into the relation (8), it works out as

$$N_m = \frac{4\pi V}{h^3} \int\limits_0^{p_*} \frac{p^2 dp}{\left(1+\beta^2 p^2\right)} = \frac{4\pi V}{h^3 \beta^3} \Big(\beta p_* - \arctan[\beta p_*]\Big) \,. \tag{9}$$

For the low energy regime with $\beta p_* \ll 1$ ($p_* \ll p_{\rm Pl}$), the relation (9) for the number of microstates behaves as $N_m \sim p_*^3$ which shows that it correctly coincides with the standard result of the non-deformed case in this regime. For the high energy regime $\beta p_* \sim 1$ ($p_* \sim p_{\rm Pl}$), however, the number of microstates (9) behaves linearly with p_* as $N_m \sim p_*$ (see also Fig. 1, where the number of microstates (9) versus p_* is plotted and also it is compared with the standard non-deformed case). This result shows that, at the high energy regime when the quantum gravity (minimal length) effects dominate, the number of microstates will be drastically decreased. Since the number of microstates for a standard one-dimensional particle (with two dimensional phase space) behaves linearly with the momentum, one could conclude that two degrees of freedom will be frozen at the high energy regime and therefore a particle has only one degree of freedom in this regime. This result suggests an effective dimensional reduction of the space from 3 to 1 dimension at the high energy regime in Snyder model. Although this result is qualitatively obtained here for a general system (without specifying a Hamiltonian function and ensemble density), we will explicitly justify this result in the next section for the particular case of an ideal gas in canonical ensemble in the light of the well-known equipartition theorem of energy.

2.3. Partition function

In order to study statistical mechanics in this framework, we generalize this setup to a many-particle system. Consider a statistical system consisting of N particles. The corresponding kinematical phase space can be obtained by the direct coupling of the single-particle phase spaces as

$$\Gamma_{\rm tot} = \Gamma_1 \times \ldots \times \Gamma_N \,, \qquad \omega_{\rm tot} = \sum_{\alpha=1}^{N} \omega_\alpha \,, \tag{10}$$

where ω_α is the symplectic structure on the phase space of the α-th particle, Γ_α. Substituting the symplectic structure (10) into the definition (5), the corresponding $6N$-dimensional Liouville volume will be

$$\omega^{3N} = \frac{1}{(3N)!} \left(\sum_{\alpha=1}^{N} \omega_\alpha\right) \wedge \ldots \wedge \left(\sum_{\alpha=1}^{N} \omega_\alpha\right) = \omega_1^3 \wedge \ldots \wedge \omega_N^3 \,, \tag{11}$$

where ω_α^3 is the six-dimensional Liouville volume corresponding to the α-th particle phase space and we have also used the fact that $\omega_\alpha^i = 0$ for $i > 3$, with ω_α^i being the i-th component of the α-th particle's Liouville volume. The quantum gravity parameter β is universal and it will be the same for all the particles. Therefore, the symplectic structure for all of the particles in Snyder-deformed phase space is given by (1) and the Liouville volume is then given by (6). Substituting the Liouville volume (6) for all of the particles in the relation (11) gives

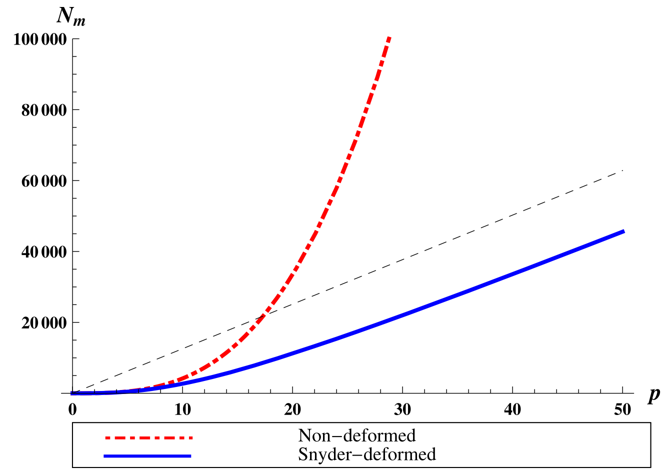

Fig. 1. The number of microstates versus the momentum p_* for a single-particle phase space. The blue solid and red dot-dashed lines represent the number of microstates in the Snyder-deformed and non-deformed phase spaces respectively. Clearly, these two curves coincide at the low energy regime $\beta p_* \ll 1$ ($p_* \ll p_{\rm Pl}$) and the deviation arises at the high energy regime $\beta p_* \sim 1$ ($p_* \sim p_{\rm Pl}$). The number of microstates in the Snyder-deformed case effectively behaves like a one-dimensional single-particle phase space (the black dashed line) which shows that two degrees of freedom freeze at the high energy regime due to the quantum gravity (minimal length) effects. This result suggests an effective dimensional reduction of the space from 3 to 1 dimension at the high energy regime. The figure is plotted for $\beta = \frac{\beta_0}{p_{\rm Pl}} = 0.1$ with $\beta_0 = 1$ and $p_{\rm Pl} = 10$.

$$\omega^{3N} = \frac{d^{3N}q\, d^{3N}p}{\left(1+\beta^2 p^2\right)^N} \,, \tag{12}$$

where clearly the standard volume $d^{3N}q\, d^{3N}p$ for the non-deformed $6N$-dimensional phase space could be recovered in the low energy limit $\beta \to 0$. While the non-deformed phase space volume assigns a uniform probability distribution over the set of microstates, the Snyder-deformed phase space volume (12) assigns a nonuniform probability distribution at the high energy regime such that the microstates with higher energy are less probable. More precisely, in the absence of any extra information for the system, the Laplace principle of indifference states that all the microstates are equally likely [26]. However, in the presence of a minimal length, as an extra information for the system, the sufficient condition for the Laplace's indifference principle is no longer satisfied in the Snyder model.

The Liouville volume (12) is invariant under the time evolution of the system on $\Gamma_{\rm tot}$ which can be easily deduced from the relation (7). Thus the Liouville theorem is satisfied on the phase space of the N-particle system $\Gamma_{\rm tot}$ which allows us to study the statistical mechanics in this setup. The canonical partition function for a system at the temperature T in Snyder model then will be

$$\mathcal{Z}_N = \frac{1}{h^{3N} N!} \int\limits_{\Gamma_{\rm tot}} \omega^{3N} \exp\big[-H/T\big] \,, \tag{13}$$

where the Gibbs factor $1/N!$ is also considered. For a system in which the particles do not interact, the total Hamiltonian function could be decomposed as $H_{\rm tot} = \sum_{\alpha=1}^{N} H_\alpha$ and the partition function (13) simplifies to

$$\mathcal{Z}_N = \frac{\mathcal{Z}_1^N}{N!} \,, \tag{14}$$

where we have defined the single-particle partition function as

$$\mathcal{Z}_1 = \frac{1}{h^3}\int_\Gamma \omega^3 \exp\big[-H/T\big] = \frac{1}{h^3}\int d^3q \int \frac{d^3p}{(1+\beta^2p^2)} \exp\big[-H(q,p)/T\big]. \quad (15)$$

Having the partition function (14) in hand, one could easily study the thermodynamical properties of the statistical systems in Snyder space.

3. Thermodynamics of ideal gas

In this section we study the thermodynamical properties of the ideal gas by means of the results obtained in previous section.

By considering an ideal gas consisting of N noninteracting particles confined in volume V at the temperature T, the corresponding single-particle partition function can be easily obtained from the relation (15) as

$$\mathcal{Z}_1[V,T] = \frac{4\pi V}{h^3}\int_0^\infty \frac{\exp\big[-\frac{p^2}{2mT}\big]p^2dp}{(1+\beta^2p^2)} = \frac{V\pi^{\frac{3}{2}}}{h^3\beta^3}\Bigg(\sqrt{2mT}\beta - \sqrt{\pi}\,\mathrm{erfc}\Big[\frac{1}{\sqrt{2mT}\beta}\Big] \times \exp\Big[\frac{1}{2mT\beta^2}\Big]\Bigg), \quad (16)$$

where $\mathrm{erfc}[x] = (2/\sqrt{\pi})\int_x^\infty e^{-t^2}dt$ is the complementary error function and also we have substituted $H = p^2/2m$ for the Hamiltonian function. To understand the qualitative behavior of the partition function (16), it is useful to rewrite the partition function in terms of the thermal wave length $\lambda = \frac{h}{\sqrt{2\pi mT}}$ as

$$\mathcal{Z}_1[V,\lambda] = \frac{V}{\lambda_{\mathrm{Pl}}^3}\Bigg(\frac{\lambda_{Pl}}{\lambda} - \sqrt{\pi}\,\mathrm{erfc}\Big[\frac{\lambda}{\lambda_{\mathrm{Pl}}}\Big]\exp\Big[\frac{\lambda^2}{\lambda_{\mathrm{Pl}}^2}\Big]\Bigg), \quad (17)$$

where we have substituted $\beta = \beta_0 l_{\mathrm{Pl}} = \frac{\beta_0}{T_{\mathrm{Pl}}}$ and also we have defined *Planck scale thermal de Broglie wavelength*

$$\lambda_{\mathrm{Pl}} = (\sqrt{2}\beta_0) \times \frac{h}{\sqrt{2\pi m_{\mathrm{Pl}}T_{\mathrm{Pl}}}} = 2\sqrt{\pi}\beta_0 l_{\mathrm{Pl}}, \quad (18)$$

where as we have mentioned before $\beta_0 = \mathcal{O}(1)$ should be fixed by experiment [25]. Expanding partition function (17) for both the low and high temperature regimes gives

$$\mathcal{Z}_1[V,\lambda] = \begin{cases} \frac{V}{\lambda^3} & \lambda \gg \lambda_{\mathrm{Pl}}, \\ \frac{V}{\lambda\lambda_{\mathrm{Pl}}^2} & \lambda \sim \lambda_{\mathrm{Pl}}. \end{cases} \quad (19)$$

As it is clear from the above relation, quantum gravity (minimal length) effects are negligible at the low temperature limit $\lambda \gg \lambda_{\mathrm{Pl}}$ $(T \ll T_{\mathrm{Pl}})$ and the standard result for the partition function of the ideal gas is recovered. Interestingly, the high-temperature behavior of the partition function (19) shows that two degrees of freedom will be frozen at the Planck scale $\lambda \sim \lambda_{\mathrm{Pl}}$ and there is only one degree of freedom for a particle in this regime. This feature, as we will show, leads to the effective reduction of the dimension of space at the high temperature regime.

From the relation (14), the total partition function for the ideal gas in Snyder space will be

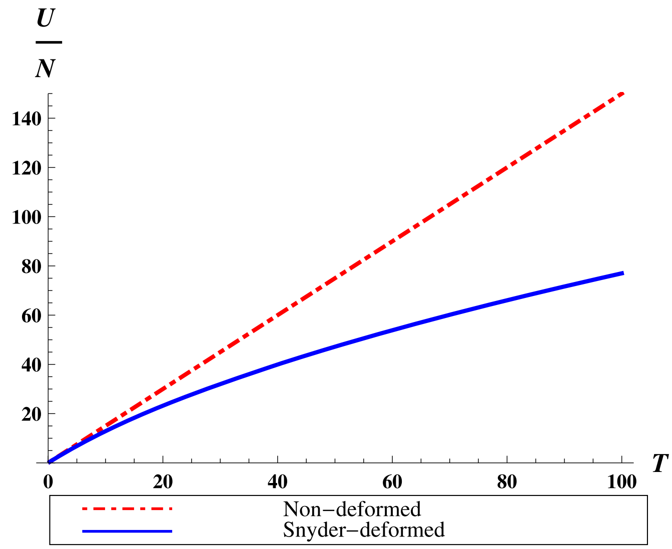

Fig. 2. Internal energy versus temperature. The blue solid and red dot-dashed lines represent the internal energy in Snyder-deformed and non-deformed phase spaces respectively. As the temperature increases, the quantum gravity (minimal length) effects become more and more appreciable.

$$\mathcal{Z}_N[V,\lambda] = \frac{\big(V/\lambda_{\mathrm{Pl}}^3\big)^N}{N!}\Bigg(\frac{\lambda_{\mathrm{Pl}}}{\lambda} - \sqrt{\pi}\,\mathrm{erfc}\Big[\frac{\lambda}{\lambda_{\mathrm{Pl}}}\Big]\exp\Big[\frac{\lambda^2}{\lambda_{\mathrm{Pl}}^2}\Big]\Bigg)^N, \quad (20)$$

from which all of the thermodynamical quantities could be derived through the standard definitions.

3.1. Internal energy and specific heat

The Helmholtz free energy F is defined as

$$F = -T\ln\big[\mathcal{Z}_N[V,\lambda]\big] = -NT\Bigg\{1 + \ln\Bigg[\frac{V}{N\lambda_{\mathrm{Pl}}^3}\Bigg(\frac{\lambda_{\mathrm{Pl}}}{\lambda} - \sqrt{\pi}\,\mathrm{erfc}\Big[\frac{\lambda}{\lambda_{\mathrm{Pl}}}\Big]\exp\Big[\frac{\lambda^2}{\lambda_{\mathrm{Pl}}^2}\Big]\Bigg)\Bigg]\Bigg\}, \quad (21)$$

where we have used the Stirling's formula $\ln[N!] \approx N\ln[N] - N$ for large N.

From the modified Helmholtz free energy (21), the internal energy for the ideal gas gets modified as

$$U = -T^2\left(\frac{\partial}{\partial T}\left(\frac{F}{T}\right)\right)_{N,V} = NT\left\{\left(2 - \sqrt{\pi}\left(\frac{\lambda}{\lambda_{\mathrm{Pl}}}\right)\mathrm{erfc}\Big[\frac{\lambda}{\lambda_{\mathrm{Pl}}}\Big]\exp\Big[\frac{\lambda^2}{\lambda_{\mathrm{Pl}}^2}\Big]\right)^{-1} - \frac{\lambda^2}{\lambda_{\mathrm{Pl}}^2}\right\}. \quad (22)$$

The internal energy versus the temperature is plotted in Fig. 2. As it is clear from the figure, while the internal energy of the standard ideal gas increases linearly with the temperature as $U_0 = \frac{3}{2}NT$, which is shown by the red dot-dashed line in the figure, the internal energy increases with a decreasing rate at the high temperature regime (the blue solid line in the figure) where the associated thermal de Broglie wavelength approaches to the Planck length l_{Pl}. Expanding the relation (22) for both low and high temperature regimes gives

$$U = \begin{cases} \frac{3}{2}NT & \lambda \gg \lambda_{\mathrm{Pl}}, \\ \frac{1}{2}NT & \lambda \sim \lambda_{\mathrm{Pl}}. \end{cases} \quad (23)$$

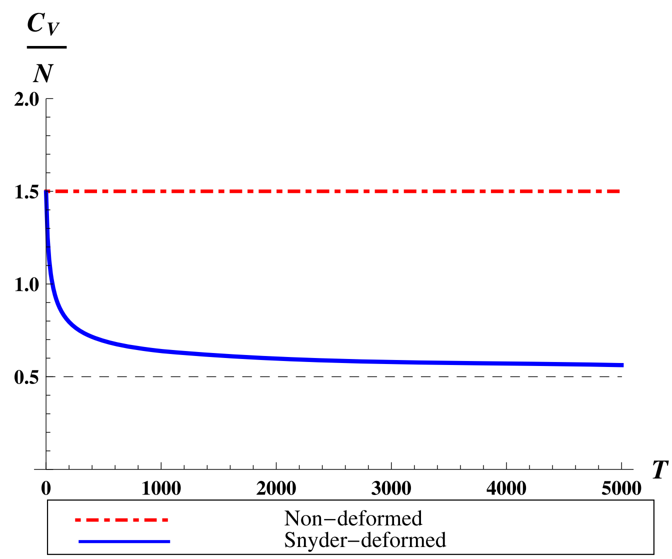

Fig. 3. Specific heat versus the temperature. The blue solid and red dot-dashed lines represent the specific heat in Snyder-deformed and non-deformed phase spaces respectively. While the specific heat is independent of the temperature as $\frac{3}{2}N$ for the case of standard ideal gas, it becomes temperature-dependent at the high energy regime in Snyder model. It asymptotically leads to the value $\frac{1}{2}N$ at the very high temperature regime which signals the effective dimensional reduction of space from 3 to 1 dimension in this setup. It is also clear from the figure that the specific heat is bounded as $\frac{1}{2} \leq \frac{C_V}{N} \leq \frac{3}{2}$ in Snyder model.

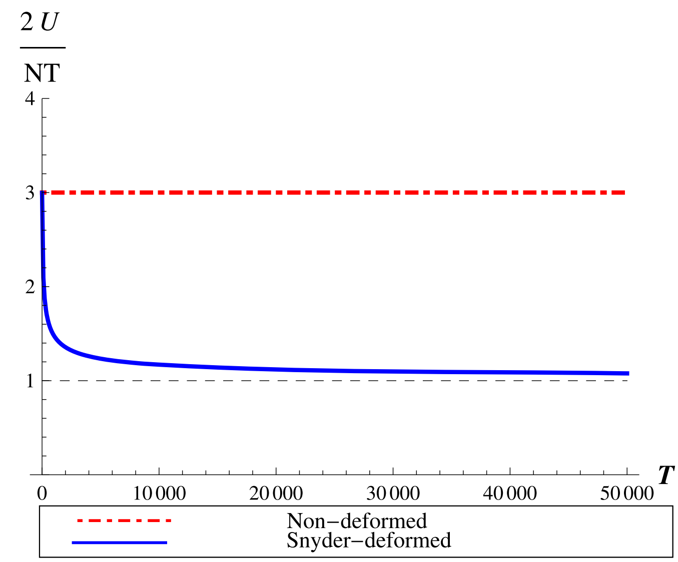

Fig. 4. Number of degrees of freedom versus the temperature. The classical equipartition theorem of energy states that the number of degrees of freedom for a particle is given by $\frac{(U/N)}{(T/2)}$. Thus, as it is clear from the figure, the number of degrees of freedom will be reduced from 3 to 1 at the high temperature regime in Snyder space. This result suggests an effective dimensional reduction of space from 3 to 1 dimension.

The high temperature behavior of the internal energy in Snyder model could be also more precisely understood from the specific heat that is defined as

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{N}{2}\,\frac{2\left(2+\frac{\lambda^2}{\lambda_{Pl}^2}\right) - \sqrt{\pi}\left(\frac{\lambda}{\lambda_{Pl}}\right)\left(3+2\frac{\lambda^2}{\lambda_{Pl}^2}\right)\mathrm{erfc}\left[\frac{\lambda}{\lambda_{Pl}}\right]\exp\left[\frac{\lambda^2}{\lambda_{Pl}^2}\right]}{\left(2-\sqrt{\pi}\left(\frac{\lambda}{\lambda_{Pl}}\right)\mathrm{erfc}\left[\frac{\lambda}{\lambda_{Pl}}\right]\exp\left[\frac{\lambda^2}{\lambda_{Pl}^2}\right]\right)^2}. \tag{24}$$

The temperature-dependent behavior of this quantity is plotted in Fig. 3. Expanding the specific heat (24) for low and high temperature regimes gives

$$C_V = \begin{cases} \frac{3}{2}N & \lambda \gg \lambda_{Pl}, \\ \frac{1}{2}N & \lambda \sim \lambda_{Pl}, \end{cases} \tag{25}$$

which shows that the specific heat approaches to $C_V \to N/2$ at the high temperature limit (see also Fig. 3). As it is clear from Fig. 3, the specific heat is bounded as $\frac{1}{2} \leq \frac{C_V}{N} \leq \frac{3}{2}$ in this setup.

In order to understand the reduction of the number of degrees of freedom in a more precise manner, we invoke the well-known theorem of *equipartition of energy* which states that each number of degree of freedom makes a contribution of $\frac{1}{2}T$ towards the internal energy and $\frac{1}{2}$ towards the specific heat. From the relations (23) and (25), it is clear that the number of degrees of freedom for the ideal gas consisting of N noninteracting particles which move on three-dimensional Euclidean space $\mathbf{R}^3$ (with $\mathbf{R}^{3N}$ configuration space), will be reduced from $3N$ to N at the high temperature regime when the thermal de Broglie wavelength of the system becomes of the order of the Planck length $\lambda \sim l_{Pl}$. In other words, there is one degree of freedom for a particle at such a high temperature regime and two degrees of freedom will be frozen due to the quantum gravity (minimal length) effects. See also Fig. 4 which shows the temperature-dependent behavior of the number of degrees freedom $\frac{(U/N)}{(T/2)}$ for a particle in Snyder model. Therefore, according to the equipartition theorem, the number of degrees of freedom for a particle is reduced from 3 to 1 at the high temperature regime in the Snyder space. This result suggests an effective high temperature dimensional reduction of space from 3 to 1 dimension in the Snyder model. Similar results have been obtained in the context of the other phenomenological approaches to minimal length scenario. For instance, in the context of the doubly special relativity theories, it is shown that the total number of degrees of freedom will be finite at the high temperature regime which shows that all of the degrees of freedom will be frozen in this setup [22]. The associated phase space then will be compact (with finite Liouville volume) [27] and the corresponding Hilbert space is also finite dimensional [28] (see Ref. [29] for some cosmological consequences of such a framework). Furthermore, in the context of polymer quantization, it is shown that the energy density of the photon gas will be proportional to $T^{5/2}$ rather than the standard Stephan–Boltzmann law that states the energy density will be proportional to T^4 [20]. This result also suggests an effective reduction to 1.5 dimensional space at the Planck scale. Thus, the dimensional reduction at the Planck scale seems to be a common feature of quantum gravity proposal [11].

3.2. Pressure and equation of state

The thermal pressure could be obtained from the Helmholtz energy (21) as

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{NT}{V}, \tag{26}$$

which shows that the equation of state relation preserves its standard form $PV = NT$ in this setup. From the relations (22) and (26), the equation of state parameter $w = \frac{P}{U/V}$ works out to be

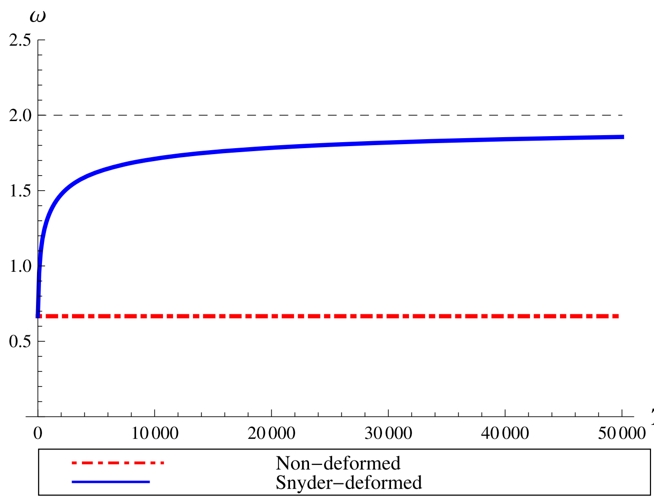

Fig. 5. Equation of state parameter versus the temperature. While the equation of state parameter is constant as $w = 2/3$ for the standard ideal gas, it turns out to be temperature-dependent quantity in Snyder space. As it is clear from the figure, it approaches to $w = 2$ at the very high temperature regime when the quantum gravity (minimal length) effects dominate. The figure shows also that $\frac{2}{3} \leq w \leq 2$.

$$w = \left\{ \left(2 - \sqrt{\pi} \left(\frac{\lambda}{\lambda_{\rm Pl}} \right) \mathrm{erfc}\left[\frac{\lambda}{\lambda_{\rm Pl}} \right] \exp\left[\frac{\lambda^2}{\lambda_{\rm Pl}^2} \right] \right)^{-1} - \frac{\lambda^2}{\lambda_{\rm Pl}^2} \right\}^{-1}. \tag{27}$$

The temperature behavior of the equation of state parameter (27) is plotted in Fig. 5. The equation of state parameter is constant as $w = \frac{2}{3}$ for the standard ideal gas. But, as it is clear from Fig. 5, it becomes temperature-dependent in Snyder model at the high temperature regime. The standard result $w = \frac{2}{3}$ can be recovered at the low temperature regime $\lambda \gg l_{\rm Pl}$ and also $w \to 2$ at the high temperature regime $\lambda \sim l_{\rm Pl}$. Thus, generally we have $\frac{2}{3} \leq w \leq 2$ which could be seen from Fig. 5 (see also Refs. [20] and [30] where the same results are obtained in the contexts of polymer quantization and holography respectively).

3.3. Entropy

As the final thermodynamical quantity, one could obtain the modification to the entropy from the minimal length effects. The entropy, however, is directly related to the number of accessible microstates and it is then natural to expect that the entropy increases with a rate smaller than the standard non-deformed case at the high temperature regime since we have shown that the number of accessible microstates will be decreased at the high temperature regime in Snyder space. The direct calculation of the entropy justifies this claim. From the Helmholtz free energy (21), the entropy of the ideal gas will be

$$\begin{aligned} \frac{S}{N} &= -\left(\frac{\partial F}{\partial T} \right)_{N,V} \\ &= 1 + \ln\left[\frac{V}{N\lambda_{\rm Pl}^3} \left(\frac{\lambda_{\rm Pl}}{\lambda} - \sqrt{\pi}\, \mathrm{erfc}\left[\frac{\lambda}{\lambda_{\rm Pl}} \right] \exp\left[\frac{\lambda^2}{\lambda_{\rm Pl}^2} \right] \right) \right] \\ &\quad + \left(2 - 2\sqrt{\pi} \left(\frac{\lambda}{\lambda_{\rm Pl}} \right) \mathrm{erfc}\left[\frac{\lambda}{\lambda_{\rm Pl}} \right] \exp\left[\frac{\lambda^2}{\lambda_{\rm Pl}^2} \right] \right)^{-1} - \frac{\lambda^2}{\lambda_{\rm Pl}^2}. \end{aligned} \tag{28}$$

In Fig. 6, the temperature-dependent behavior of the ideal gas entropy (28) is plotted. As it is clear from the figure, in contrast to the non-deformed case, the entropy increases with smaller rate in the Snyder model. This result is also a direct consequence of the effective dimensional reduction of space from three to one dimension for a particle in the Snyder space.

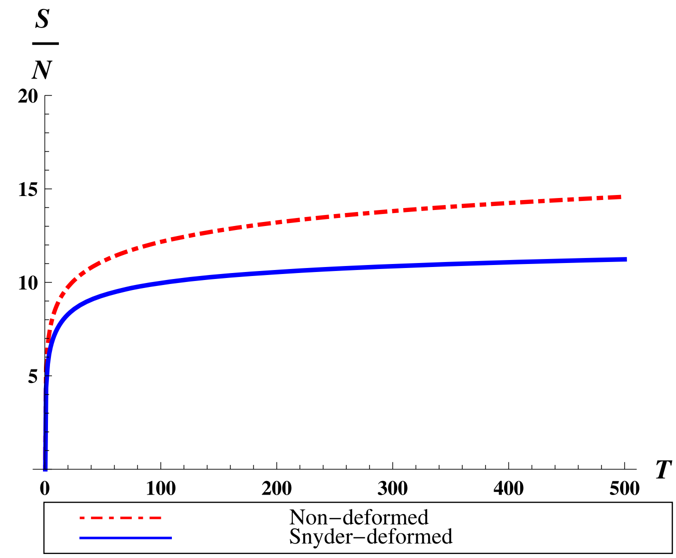

Fig. 6. Entropy versus the temperature. It is clear that, in the Snyder model the entropy increases with a rate smaller than the standard non-deformed case. This is because of the fact that the number of accessible microstates is decreased at the high temperature regime due to the quantum gravity (minimal length) effects.

Although, as we have shown, the minimal length effects will become important only at the very high temperature regime when the thermal de Broglie wavelength of the system becomes of the order of the Planck length, it is also useful to consider the low temperature behavior in order to estimate the order of the magnitude of the quantum gravity effects on the thermodynamical quantities of the ideal gas. It is straightforward to show that the first order quantum gravity corrections to all of the thermodynamical quantities such as the internal energy (22), specific heat (24), and entropy (28) are proportional to $(\lambda_{\rm Pl}/\lambda)^2$. Therefore, the thermal de Broglie wavelength λ is an appropriate parameter to determine when quantum gravitational effects will become significant, much similar in the same way as pure quantum mechanical effects become important in the standard statistical mechanics. Indeed, the pure quantum mechanical effects will become important when the thermal de Broglie wavelength becomes of the order of the mean interparticle distance $(V/N)^{1/3}$. Similarly and much in the same way, quantum gravitational (minimal length) effects will become important when the thermal de Broglie wavelength of the system becomes of the order of the Planck length.

4. Summary and conclusions

Existence of a universal minimal length, preferably of the order of the Planck length, is a common address of quantum gravity candidates such as string theory and loop quantum gravity. Beside, this issue could be achieved from the spaces with deformed structures. In this paper, we formulated the statistical mechanics in Snyder space in the semiclassical regime. Existence of a minimal length, as an extra information for the system under consideration, significantly changes the probability distribution over the set of microstates in this setup. We obtained the corresponding deformed invariant Liouville volume which determines the number of accessible microstates for a statistical system and we have found that 2/3 of the degrees of freedom will be frozen at the high energy regime. Generalizing the setup into a many-particle system, we then obtained the modified partition function for the ideal gas in the Maxwell–Boltzmann statistics by means of the deformed Liouville volume and we have calculated the associated thermodynamical quantities such as the internal energy, specific heat, equation of state parameter, and entropy in the Snyder space. The results show that at the high temperature regime, when the thermal de Broglie wavelength becomes of the order of the Planck length, the quantum gravity (minimal length) effects dominate which significantly change the thermodynamical properties of the ideal gas. Invoking the equipartition theorem of energy, we explicitly showed that 2/3 of the number of degrees of freedom will be frozen at the high temperature regime for the special case of the ideal gas which also confirms our previous claim. This result suggests an effective dimensional reduction of the space from three to one dimension at the high temperature regime which is also a common feature in alternative approaches to quantum gravity proposal. Also, our analysis shows that $\frac{1}{2} \leq \frac{C_V}{N} \leq \frac{3}{2}$ and $\frac{2}{3} \leq w \leq 2$ for the specific heat and the equation of state parameter respectively which are the direct consequences of the effective dimensional reduction of the space at the Planck scale. Although the quantum gravity effects would become important only at the high temperature regime, considering the low temperature limit is useful to estimate the order of the magnitude of the quantum gravity corrections to the thermodynamical quantities. Evidently, the first order corrections to the internal energy, specific heat, and entropy are of the order of $(\lambda_{\rm Pl}/\lambda)^2$, where $\lambda_{\rm Pl}$ is the Planck scale thermal de Broglie wavelength (18) and λ is the standard thermal de Broglie wavelength.

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